Ellipsoidal Mirror Analysis
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Abstract—Presented here are results of an analysis of an optical system concept using elliptical mirror as the imaging element for microscopy.

Index Terms—Imaging, stigmatic image, homocentric wave, microscopy, optical system analysis, ellipsoidal mirror.

I. INTRODUCTION

OPTICAL systems for microscopy usually assemble refractive elements. Because of the aberrations of refractive elements such systems combine elements with different properties in order to optimize the aberrations. Therefore optical systems for microscopy usually result in complex designs. The ultimate goal in a optical system design is to approach the stigmatic image, where each point of the object is imaged as its corresponding point in the image. There are only a few cases where the stigmatic image is possible. The ellipsoidal mirror is one of them. The properties of a concept of an imaging system for microscopy with a single ellipsoidal mirror will be analysed.

II. STIGMATIC IMAGING

Stigmatic imaging is the case where a point of a monochromatic object transforms into a point within the image. The optical system capable of such a point-to-point imaging has to transform the homocentric spherical wave (Fig. 1) again into the homocentric spherical wave (Fig. 2). We will not assume the wave properties of the light so we can work with rays as normals to the waves.

Let’s derive the shape of the meridian of the surface capable of stigmatic imaging of point A into point A’ (Fig. 3). The geometric distances s and s’ expressed in terms of e, y, z:

\[ n\sqrt{(z+e)^2 + y^2} + n’\sqrt{(e-z)^2 + y^2} = 2a \]  
(1)

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Fig. 1. Homocentric spherical wave - diverging.

\[ 16a^2n'^2 [(e-z)^2 + y^2] = [4a^2 + n'^2 [(e-z)^2 + y^2] - n^2((z-e)^2 + y^2)]^2 \]  
(2)

Fig. 3. Derivation of the stigmatic meridian [1], which is an equation of the 4th order.

In general it is very complicated to manufacture optical elements shaped according to the 4th order equations. There are specific cases when the equation 2 changes to the 2nd order equation, which means more availability for manufacturing.

A. Ellipsoid or Hyperboloid (n’ = −n)

If n’ = −n and also n = 1 (a mirror in the air) the equation (2) becomes:

\[ \frac{z^2}{a^2} + \frac{y^2}{a^2 - e^2} = 1 \]  
(3)

If a > e, this is the equation of an ellipsis (Fig. 4) with e = \( \sqrt{a^2 - b^2} \).

Points A and A’ are geometric focal points of the ellipsis. The point A’ is a real image of the point A, which enables the ellipsis as a standalone imaging element (similar to a positive lens) and will be further discussed.
If \( a < e \) the equation (3) becomes the equation of hyperbola (Fig. 5):

\[
\frac{z^2}{a^2} - \frac{y^2}{e^2 - a^2} = 1
\]  

**B. Sphere**

If \( 2a = 0 \) the equation (2) also becomes the 2nd order equation (according to Fig. 3):

\[
n^2(z^2 + 2ez + e^2 + y^2) = n'^2(e^2 - 2ez + z^2 + y^2)
\]  

After some manipulation and introduction of new symbols we can see that the meridian is a circle (Fig. 6) and the stigmatic imaging holds for:

\[
p' = \frac{n + n'}{n'}r
\]

and

\[
p = \frac{n + n'}{n}r
\]

so that

\[
\frac{p'}{p} = \frac{n}{n'} \implies p'n' = pn
\]

Distances \( p \) and \( p' \) point to the same side of the meridian - either to the left (Fig. 8) or right (Fig. 7).

**C. Other cases**

So far we have identified several cases of stigmatic imaging for object and image located in the finite distances. There are other cases when equation (2) produces a meridian with stigmatic imaging. However these are the cases when either object or image lies in the infinity.

We will focus on the case of ellipsoidal mirror for possible application as a single imaging element for microscopy.
The analysis of the equation (2) has shown that an ellipsoidal mirror is capable of stigmatic imaging when the object lies in one geometric focal point of the ellipsoid. The image is then formed in the second geometric focal point of the ellipsoid.

This property can be exploited for imaging system according to figures 9-11.

![Ellipsoidal mirror](image)

**Fig. 9.** Ellipsoidal mirror. One box depicts the object and the second box depicts its image (i.e. the image sensor).

![Ellipsoidal mirror - top view](image)

**Fig. 10.** Ellipsoidal mirror - top view.

![Ellipsoidal mirror - front view](image)

**Fig. 11.** Ellipsoidal mirror - front view.

The location (and shape) of the aperture will have a substantial impact on the imaging. We will start with the aperture located centred around the point $C$ on top of the ellipsoid. This symmetrical placement of the aperture should provide better results than any other asymmetric placement.

For $d = 0.5$ and $\omega = 30$ deg the circle of confusion is about $q \approx 1$. This means that a single object point would get imaged as a 1(mm) large spot.

The substantial factor influencing the size of the circle of confusion $q$ is the aperture angle $\omega$. The wider the angle the larger the size of the circle of confusion. If we set a specific limit for the maximum value of $q$, say 0.01 (ten times bigger that the pixel size), we can calculate the corresponding maximal aperture angle $2\omega = 0.34$ deg (about 20 arc minutes). This is a very small aperture angle. The diameter of the corresponding aperture would have been approximately 0.3(mm).

![Raytracing the elliptical mirror](image)

**Fig. 12.** Raytracing the elliptical mirror.

So far the concept of a single element microscopy imaging system with an ellipsoidal mirror seems problematic. However the calculations were done as a quick estimate using a symbolic geometry software [2]. Another techniques for the analysis could provide more accurate estimate. Tools like [3] or [4] allow complex visualisations of aberrations of optical systems and provide tools for optimizing the design.

![Raytracing the elliptical mirror - another arrangement](image)

**Fig. 13.** Raytracing the elliptical mirror - another arrangement.

The quick estimate of the performance of the proposed concept of a single element ellipsoidal mirror imaging system for
microscopy shows that its properties make it not feasible. The future work will be focused on more accurate analysis of the concept using tools like [3] or [4]. Other possible arrangement of the concept depicted in Fig. 13 will be analysed. This arrangement could provide better performance because both object and its image are located on surfaces lying in focal planes of the ellipsoid. However this arrangement induces problems with positioning the object and the image sensor.

REFERENCES